Table 1 Comparison of Eq. (3) with data adjusted to $T_w = 0$

M	$F_c \times 10^3$	$\frac{F_c}{St_0}$	$\frac{St}{St_0}$	$\frac{T_{aw}}{T_{aw0}}$	$\frac{T_w}{T_{aw0}}$	$R(T_w)$	R_E	$R(0)$	$\epsilon_1(\%)$
2.0	0.5	0.467	0.754	0.991	1.109	0.638	0.618	0.617	0.1
2.0	1.5	1.400	0.530	0.969	1.109	0.325	0.279	0.269	1.0
3.2	0.5	0.893	0.707	0.990	1.167	0.456	0.420	0.438	−1.8
3.2	1.5	2.68	0.397	0.966	1.167	0.151	0.120	0.114	0.6

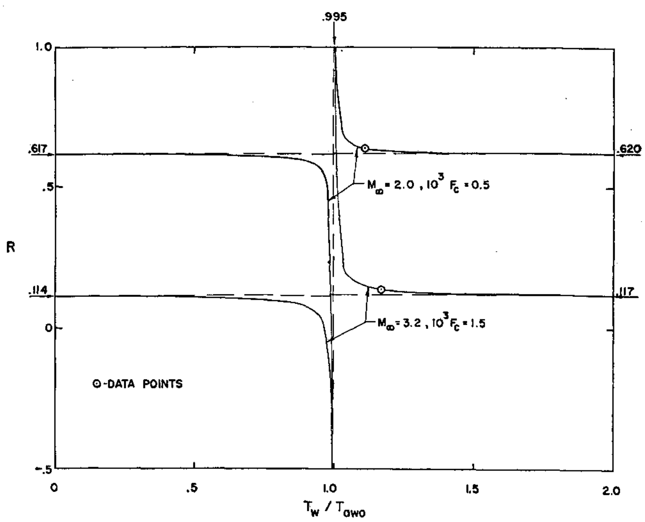

Fig. 1 The effectiveness as a function of wall temperature

0.9. Apparently also, from the example shown, the resulting R should represent well the value desired at $T_w/T_{aw0} = 0.1$. On this argument, based upon data taken with nitrogen injection, a program was planned wherein nine different coolant gases were injected. It was convenient for these latter experiments to set $T_w/T_{aw0} > 1.1$ (i.e., to preheat the "coolant" so that heat flowed from the plate to the stream). Again referring to the nitrogen injection case, one then may transcribe the data exactly to a reference value at $T_w/T_{aw0} = 0$ using Eq. (2). These points will be representative of all low T_w/T_{aw0} conditions because of the flatness of the hyperbola in this region. One may compare these transcribed data with the recommended empirical curve

$$R_E = (1 + F_c/3St_0)^{-3} \tag{3}$$

Since $R = 1$ for zero injection, one may choose unity as the proper reference quantity. Then the relative error may be taken as simply

$$\epsilon_1 = R_E - R(0) \tag{4}$$

In Table 1 these error estimates are given for four representative cases. These are comparable to, or better than, the errors accepted as tolerable in many heat transfer measurements of engineering value.

The data obtained with the various coolants scattered considerably due to the exigencies of the test situation. One would think it to be advisable for someone to repeat these measurements. Nevertheless, the results for all gases are represented well by Eq. (3), which also correlated the nitrogen data at two Mach numbers. Insofar as this formula represents all data available to date for turbulent flow over a flat plate, it properly may be called "universal," at least tentatively. If it stands the test of time, it is believed that it will be very useful in practical cases where $T_w << T_{aw0}$, despite its academically interesting singularity.

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Application of the Mangler Transformation to a Special Class of Power Law Bodies

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Nomenclature

τ = nondimensional shear stress (defined by Pai[1])
r = body radius at any point
s = surface running length
x = distance along the axis of symmetry
n = exponent in the body expression

Subscripts

A = axially symmetric
$2D$ = two-dimensional

REFINED analyses of laminar shear stress and heat transfer can be performed by application of the Mangler transformation to the specific axially symmetric body under consideration. The present note is concerned with bodies of the $r = x^n$ class.

The following expression for the ratio of laminar shear stress on an axially symmetric body to that on a two-dimensional body has been shown[1] through the application of the Mangler transformation:

$$\frac{\tau_A}{\tau_{2D}} = \left[sr^2(s) \bigg/ \int_0^s r^2(s)ds \right]^{1/2} \tag{1}$$

Let x be introduced as the distance along the axis of symmetry. Recalling that

$$ds = [1 + (dr/dx)^2]^{1/2}dx$$

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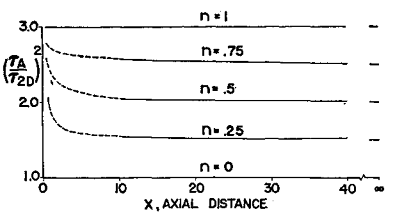

Fig. 1 Ratio of laminal shear stess for two-dimensional and axially symmetric bodies of the form $r = x^n$

and

$$s = \int_0^s \left[1 + \left(\frac{dr}{dx}\right)^2\right]^{1/2} dx$$

one can transpose Eq. (1) to the axis of symmetry as follows:

$$\frac{\tau_A}{\tau_{2D}} = \left[\frac{r^2(x) \int_0^x [1 + (dr/dx)^2]^{1/2}\, dx}{\int_0^x r^2(x)\,[1 + (dr/dx)^2]^{1/2}\, dx}\right]^{1/2} \qquad (2)$$

For the class of bodies considered in this note, namely, $r = x^n$, Eq. (2) may be expressed as

$$\left(\frac{\tau_A}{\tau_{2D}}\right)^2 = \frac{X^{2n} \int_0^x [1 + n^2x^{2(n-1)}]^{1/2}\, dx}{\int_0^x x^{2n}[1 + n^2x^{2(n-1)}]^{1/2}\, dx} \qquad (3)$$

Although direct integration of Eq. (3) is not possible (except for $n = 0, \frac{1}{2}, 1$, and 2), some results of a 16-point Gaussian numerical integration are shown in Fig. 1. [It is noted that the basic assumption that the body dimension must be much greater than the boundary layer thickness limits the accuracy of Eq. (3) at small values of x.] It is possible to examine the limits of this expression by applying l'Hospital's rule. Inspection of Eq. (3) reveals that a series expansion is divergent for values of n greater than 1.0; hence, l'Hospital's rule is valid in this work for the condition $n \leq 1$:

$$\lim_{x\to a} \left(\frac{\tau_A}{\tau_{2D}}\right)^2 =$$

$$\lim_{x\to a} \left[\frac{[1 + n^2x^{2(n-1)}]^{1/2} + (2n/x)\int_0^x [1 + n^2x^{2(n-1)}]^{1/2}\, dx}{[1 + n^2x^{2(n-1)}]^{1/2}}\right] \qquad (n \leq 1) \quad (4)$$

Rearranging Eq. (4) and applying the rule again, one obtains the following:

$$\lim_{x\to a}\left(\frac{\tau_A}{\tau_{2D}}\right)^2 - 1 = \lim_{x\to a}\left\{\frac{2n[1 + n^2x^{2(n-1)}]}{[1 + n^3x^{2(n-1)}]}\right\} \quad (n \leq 1) \quad (5)$$

This equation reduces to

$$\underset{x\to\infty}{(\tau_A/\tau_{2D})} = [1 + 2n]^{1/2} \qquad (n \leq 1) \quad (6)$$

It is important to note that Eq. (3) is an important parameter in laminar aerodynamic heating and laminar shear stress calculations but is much too cumbersome to be a useful engineering tool. Equation (6), however, is a sufficiently simple expression to be a useful tool. Also, Eq. (6) yields reasonably accurate results well into the area of doubtful validity of Eq. (3) (at small values of x).

Evaluation of both Eqs. (3) and (6) for $n = 0$ (cylinder) and $n = 1$ (cone) yields

$$\tau_A/\tau_{2D} = \underset{x\to\infty}{\tau_A/\tau_{2D}} = 1$$

and

$$\tau_A/\tau_{2D} = \tau_A/\tau_{2D} = 3^{1/2}$$

respectively. These results agree with Mangler's work.

τ_A may be determined after establishing an adequate pressure gradient (Newtonian, tangent cone, test data, etc.) and solving for τ_{2D}. This work, which is beyond the intended scope of this note, is in progress.

Circular Orbit Partial Derivatives

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Circular orbit partial derivatives are presented in both rotating and nonrotating coordinate systems for the following cutoff conditions: time, angle, and downrange distance. The accuracy of propagating errors through the partials depends on the coordinate system chosen, with the most accurate propagation being in the polar coordinate system.

I. Introduction

BALLISTIC orbit accuracy analysis involves propagation of initial errors either in the form of a state vector or in the form of a covariance matrix through a matrix of partial derivatives to determine the final error. The partial derivatives depend on the cutoff condition that is determined by the mission. Examples of cutoff conditions are time (deboost from orbit at a specified time) and downrange distance (deboost from orbit using a satellite mapping device). Preceding the presentation of the partials are discussions of different coordinate systems and the differences between rotating and nonrotating coordinate systems.

II. Coordinate Systems

A family of curvilinear orthogonal coordinate systems is illustrated in Fig. 1, where one of the axes is a straight line normal to the velocity vector, and the other axis passes through the nominal point perpendicular to the first and consists of an arc of a circle which curves inward toward the attracting body. These coordinate systems have the common feature that, about the nominal point, they are all equivalent to first order. Any first-order perturbation analysis about the nominal point therefore is independent of which one of this family is chosen and will result in the same matrix of partials. An initial error propagated in the different members of the family will result in a locus of possible final errors, all correct to the first order. An example is illustrated in Fig. 1. Since the correct final error lies close to the nominal orbit, and since the curvilinear coordinate system with radius equal to the radius of the circular orbit follows the nominal trajectory exactly, this coordinate system will result in a more accurate prediction of the final error than any of the other curvilinear coordinate systems. This coordinate system is the polar coordinate system used in Refs. 1–3.

To show analytically that this solution is identical to the Cartesian coordinate system (a limiting member of the forementioned family) used in Refs. 4–6, consider the following

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